Slope-deflection equations

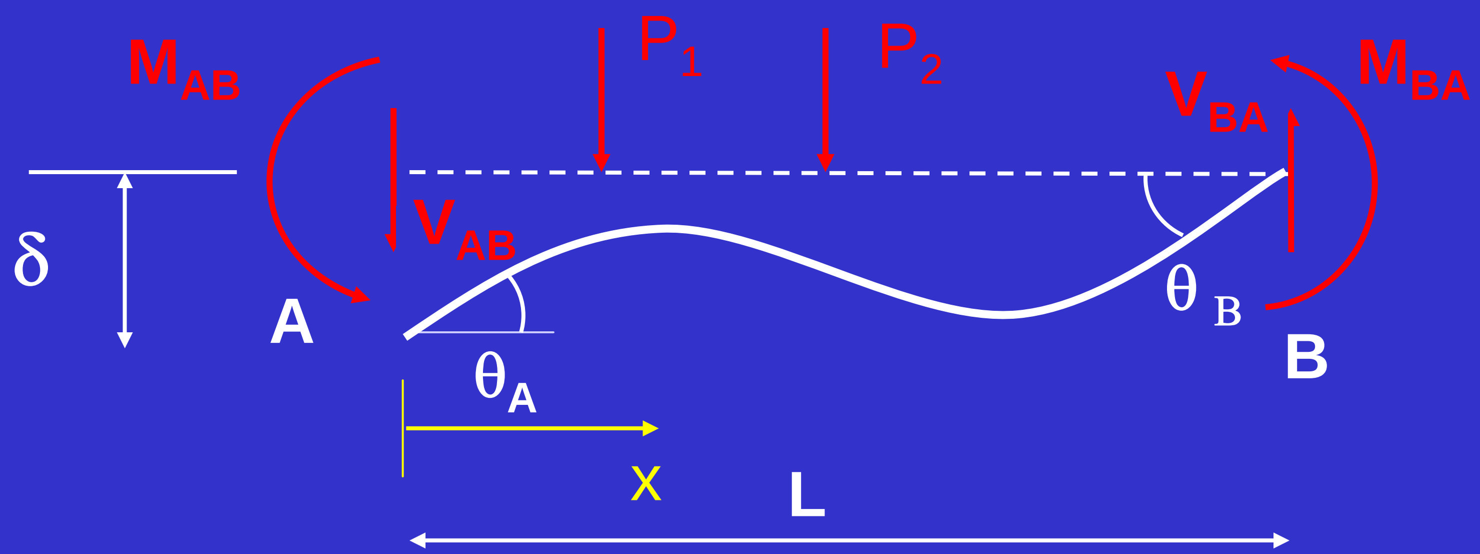
Equations can be derived relating the rotations & deflections of the ends of a beam to the end moments.

A consistent sign convention must again be adopted.

- Anti-clockwise moments are positive
- Anti-clockwise rotations are positive
- Relative displacements of beam ends are positive if the beam rotates in an anti-clockwise direction

Shear forces taken as positive if ↓↑

Slope-deflection equations



Consider a beam with applied end deflections and rotations

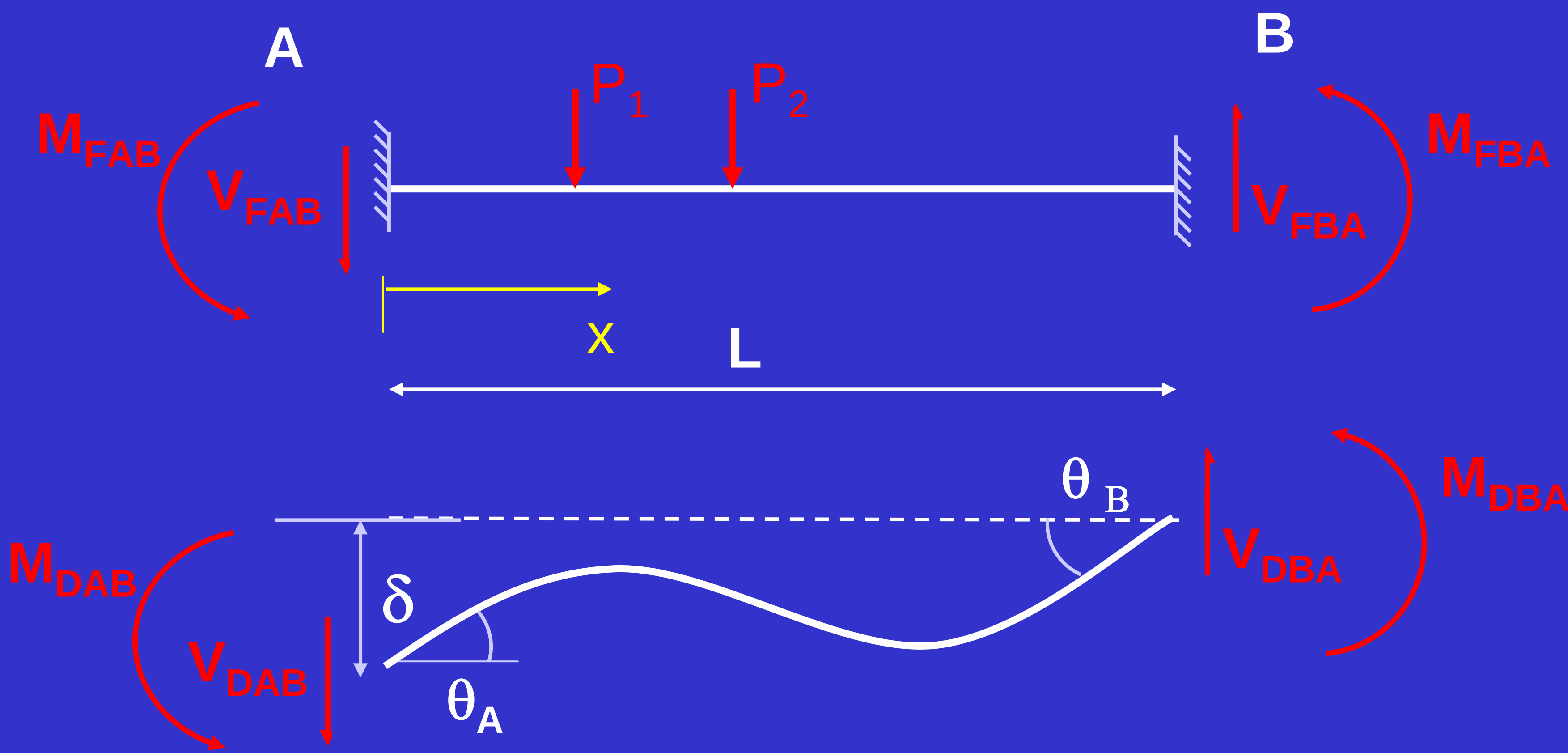
The beam also has loadings on the span

As a result of the loading and deflections/rotations, there are end moments and shear forces induced.

Slope-deflection equations

Using the Principle of Supposition, we can first analyse the beam with the load applied and with the ends fully-fixed

Then we can add the effects of the beam with no applied loading, but with end deflections and rotations



Slope-deflection equations

The total moments can be obtained by adding the fixed-end moments and the displacement moments.

$$M_{AB} = M_{FAB} + M_{DAB}$$

$$M_{BA} = M_{FBA} + M_{DBA}$$

We have already studied how to determine fixed-end moments.

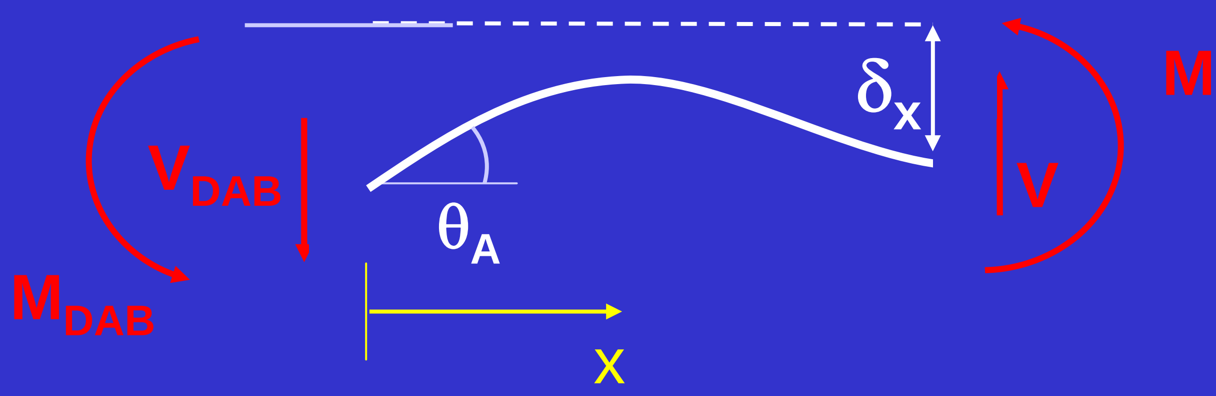
The displacement moments can now be found by deriving slope-deflection equations.

Slope-deflection equations

Considering the beam with end deflection/rotations

By taking a section at a distance x:

$$\boxed{\Sigma M = 0} \longrightarrow \boxed{M = -M_{DAB} - V_{DAB}x} \quad \boxed{= EI\frac{d^2y}{dx^2}}$$



Slope-deflection equations

$$EI\frac{d^2y}{dx^2} = -M_{DAB} - V_{DAB}x$$

Integrating once:

$$EI\frac{dy}{dx} = -M_{DAB}x - V_{DAB}\frac{x^2}{2} + EI\theta_A$$

At x = 0, dy/dx = θ_A
$\rightarrow$ A = EI θ_A

Integrating again:

$$EIy = -M_{DAB}\frac{x^2}{2} - V_{DAB}\frac{x^3}{6} + EI\theta_A x$$

At x = 0, y = 0
$\rightarrow$ B = 0

Slope-deflection equations

$$EI\theta_B = -M_{DAB}L - V_{DAB}\frac{L^2}{2} + EI\theta_A$$

At x = L, dy/dx = θ_B

$$EI\delta = -M_{DAB}\frac{L^2}{2} - V_{DAB}\frac{L^3}{6} + EI\theta_A L$$

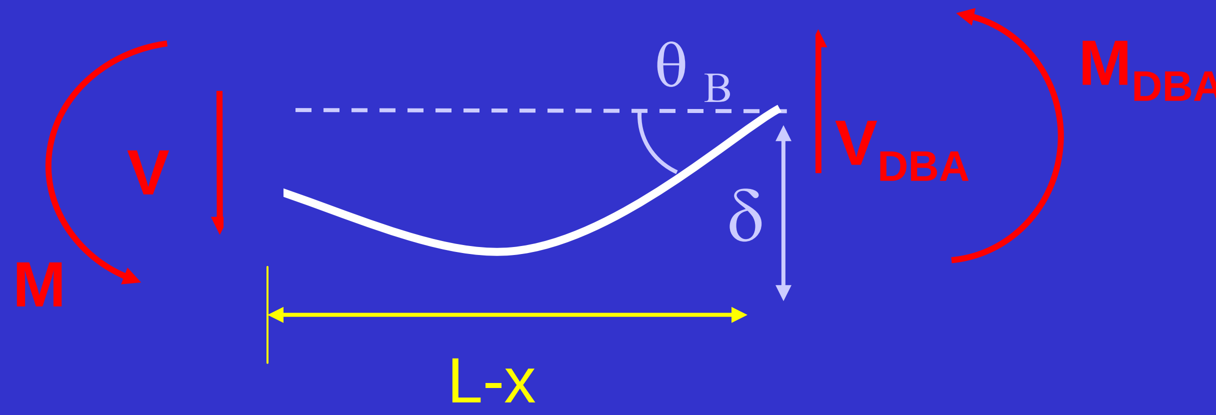
At x = L, y = δ

Eliminating V_{DAB} from these equations gives:

$$M_{DAB} = \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\delta}{L}\right)$$

Slope-deflection equations

By taking looking at the beam on the other side at a distance x:



We can go through a similar process to find:

$$M_{DBA} = \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\delta}{L}\right)$$

Slope-deflection equations

$$M_{DAB} = \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\delta}{L}\right)$$

$$M_{DBA} = \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\delta}{L}\right)$$

These are the slope-deflection equations which can be used to relate the end deflection and rotations of a beam to the end moments

We have derived the slope-deflection equations.
Now how can they be used?

1. Can be used to analyse statically indeterminate plane-frame structures

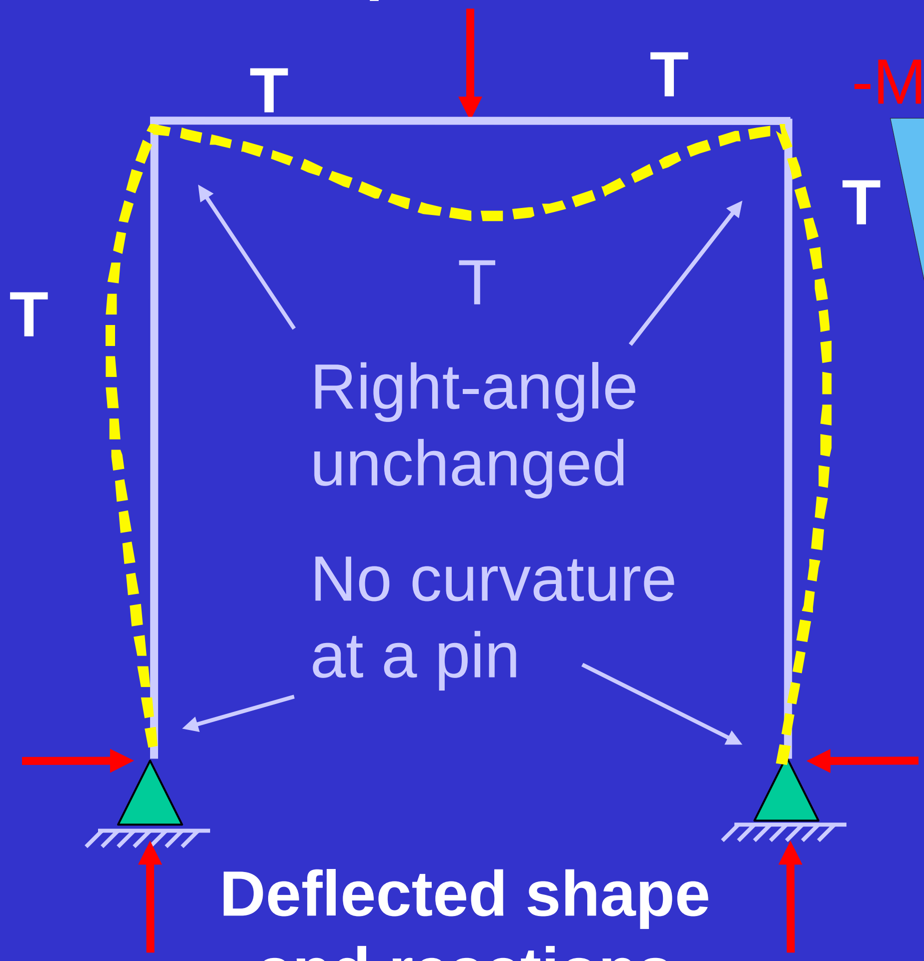
2. They are the basis of an iterative solution procedure for statically indeterminate plane-frame structures: Moment-distribution method

3. They are the basis for powerful matrix solution methods: Computer analysis

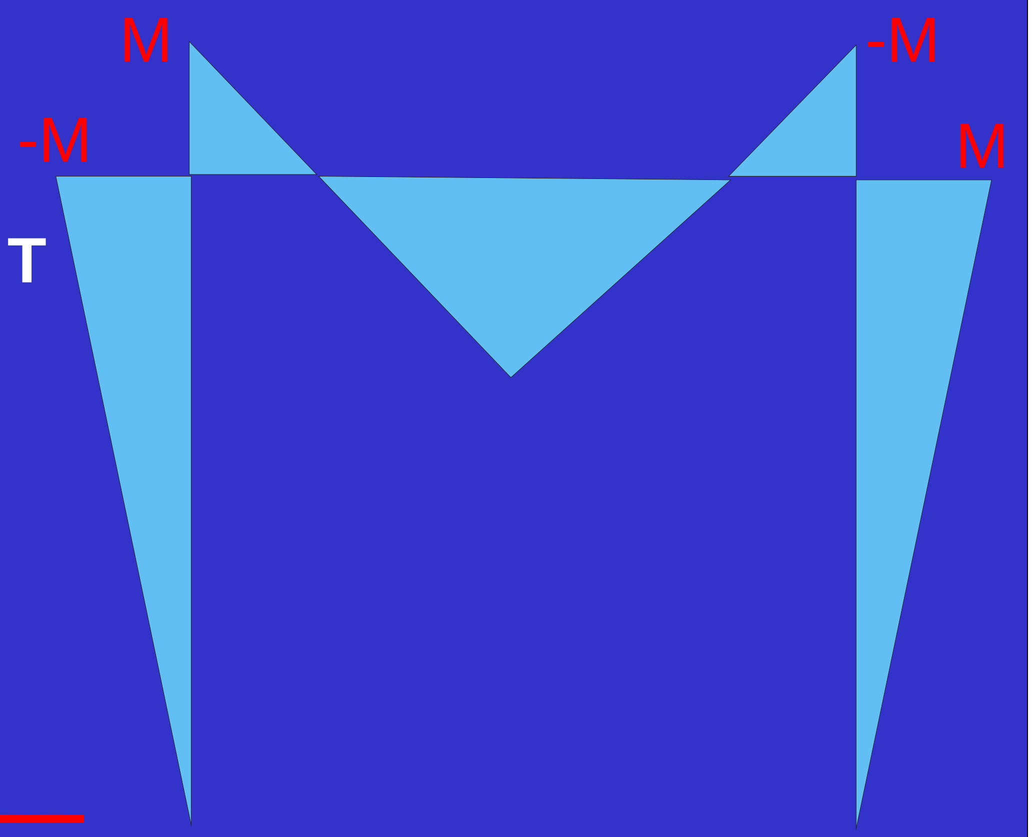
Before carrying out a numerical analysis, it is useful to sketch the expected deformed shape of the structure together with the expected reaction type and direction.

It is also helpful to sketch how you expect the bending moment diagram to look

Some examples: 1

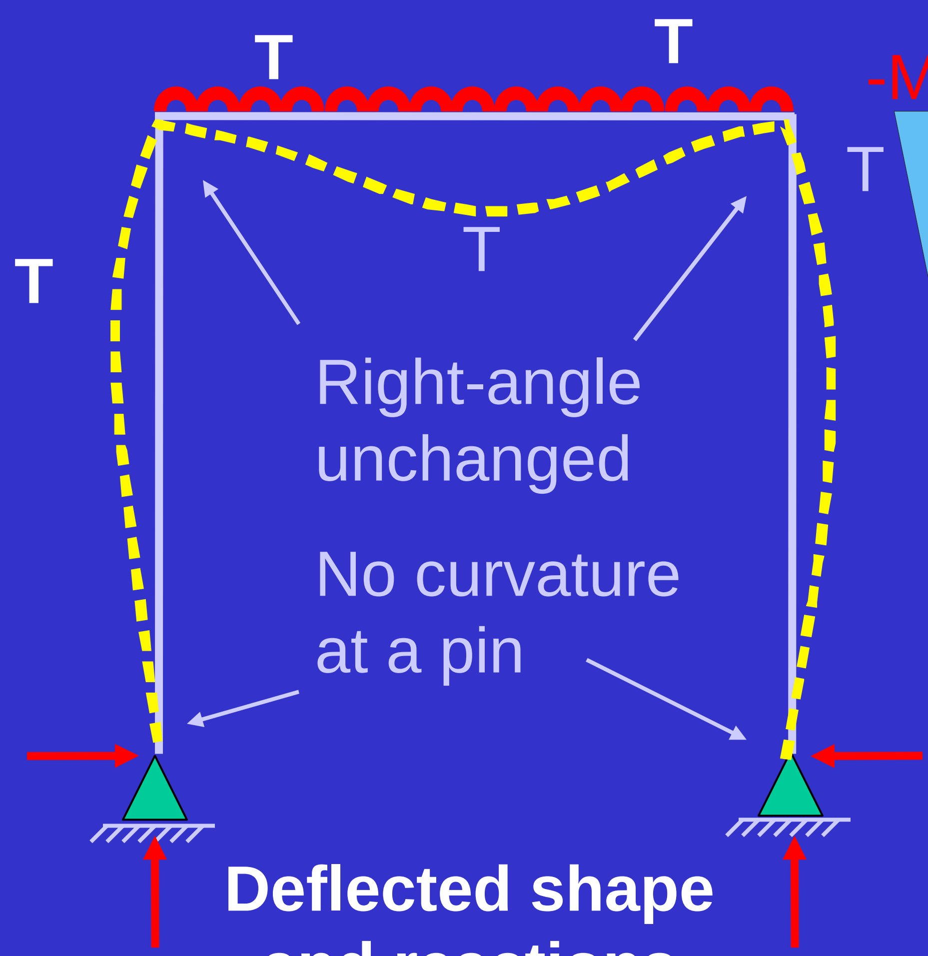


Deflected shape and reactions

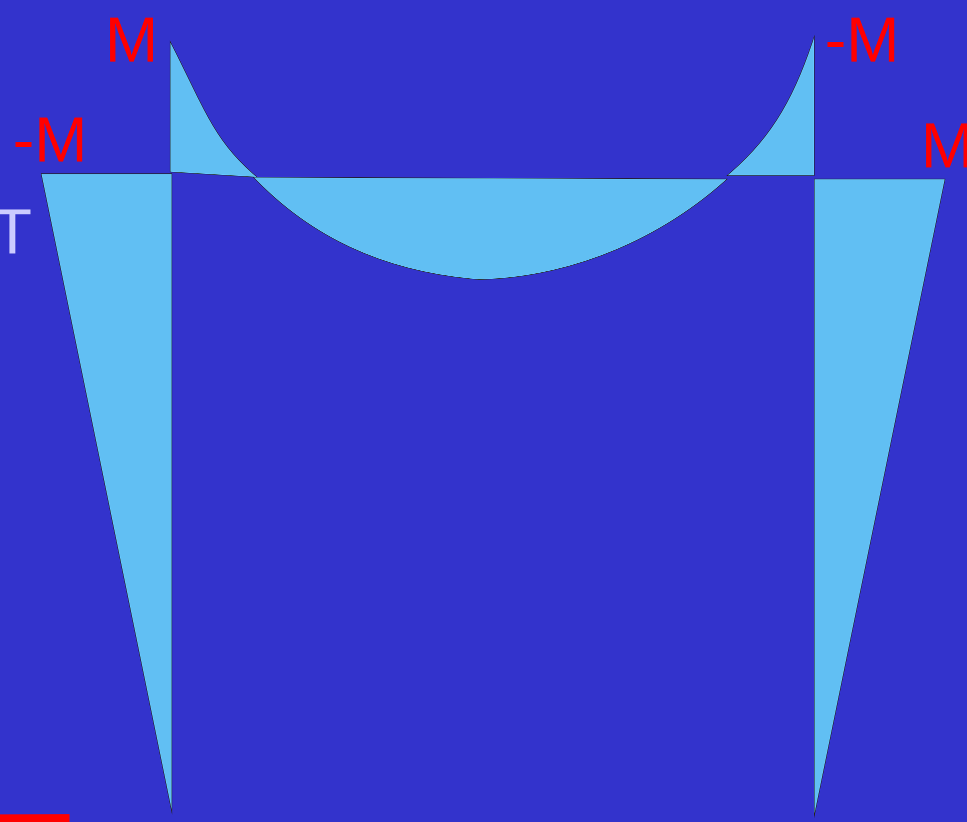


Bending moment diagram

Some examples: 2

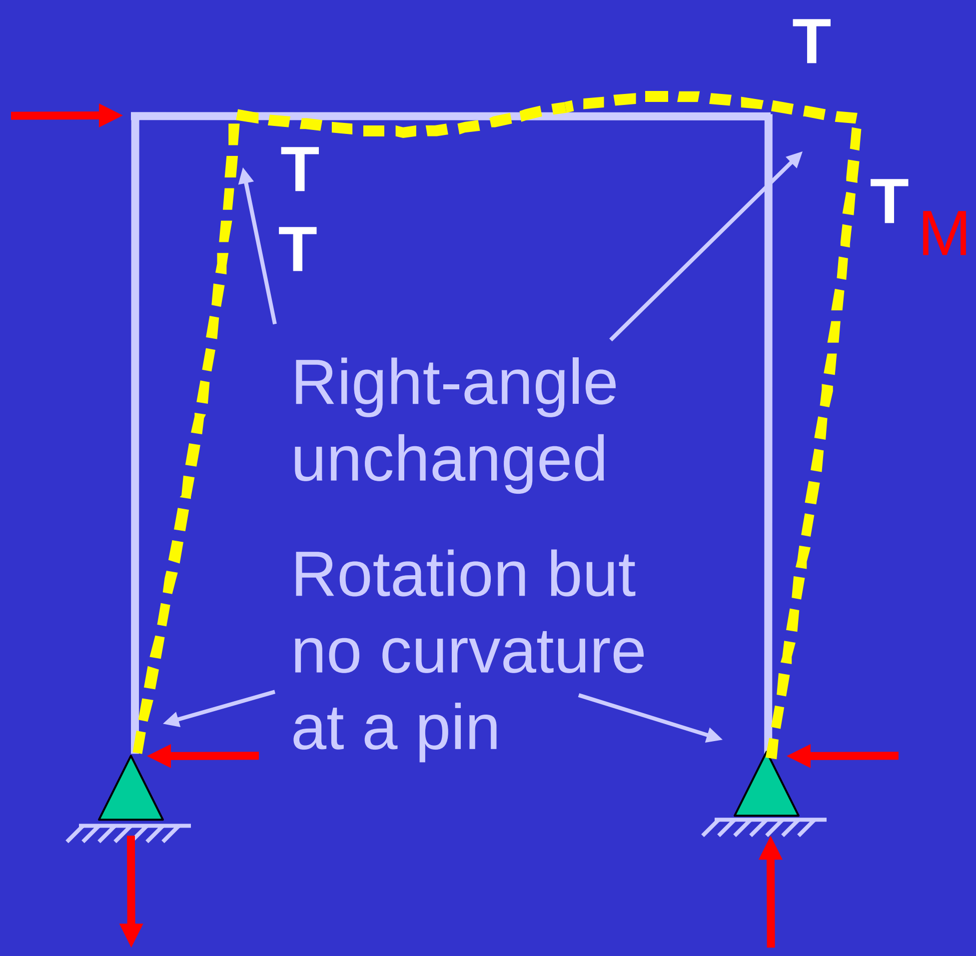


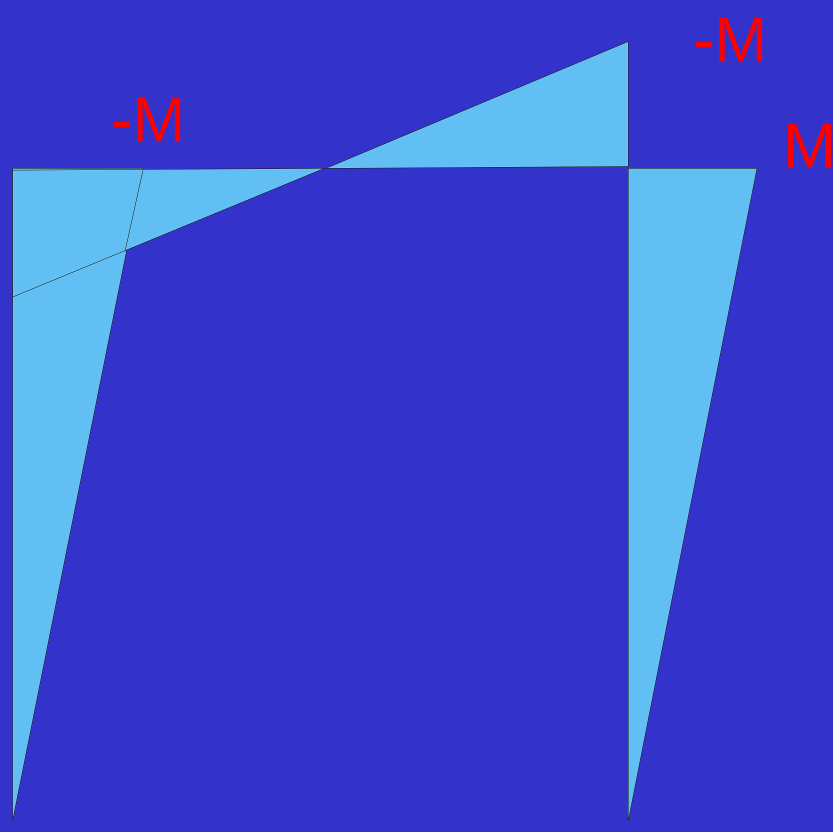
Deflected shape and reactions



Bending moment diagram

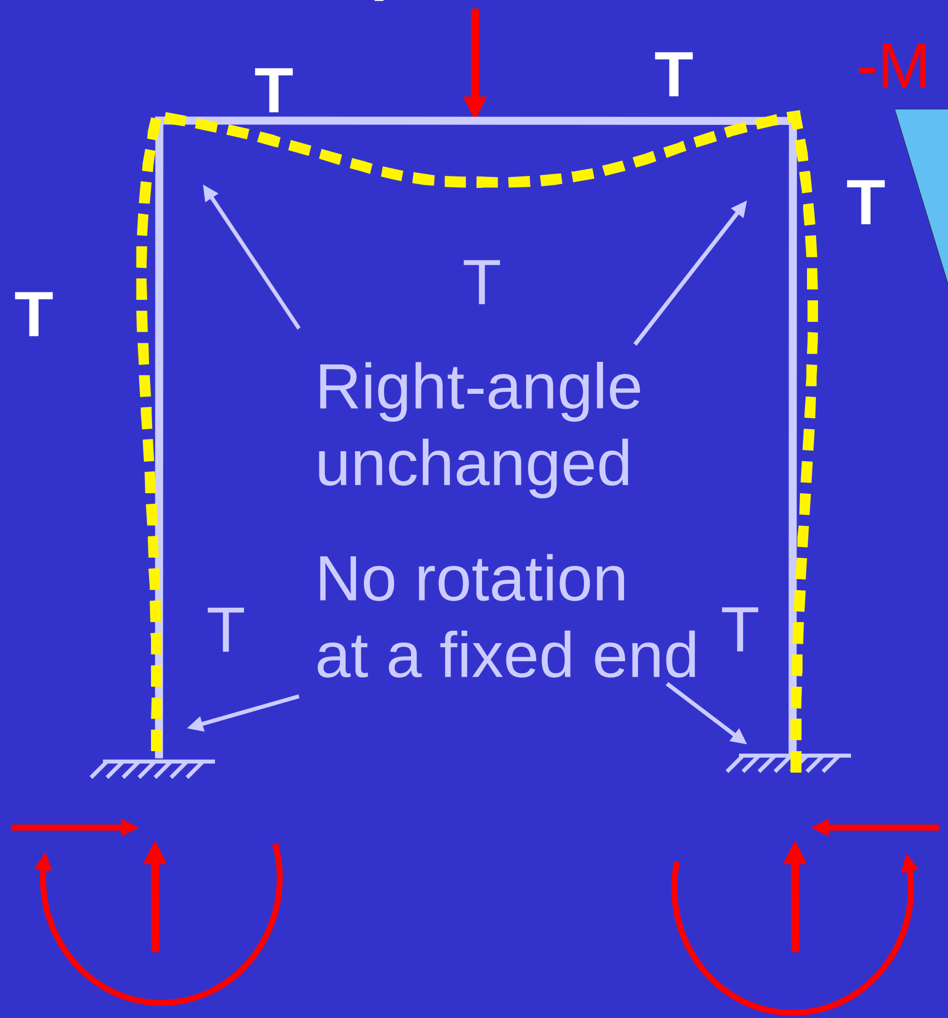
Some examples: 3

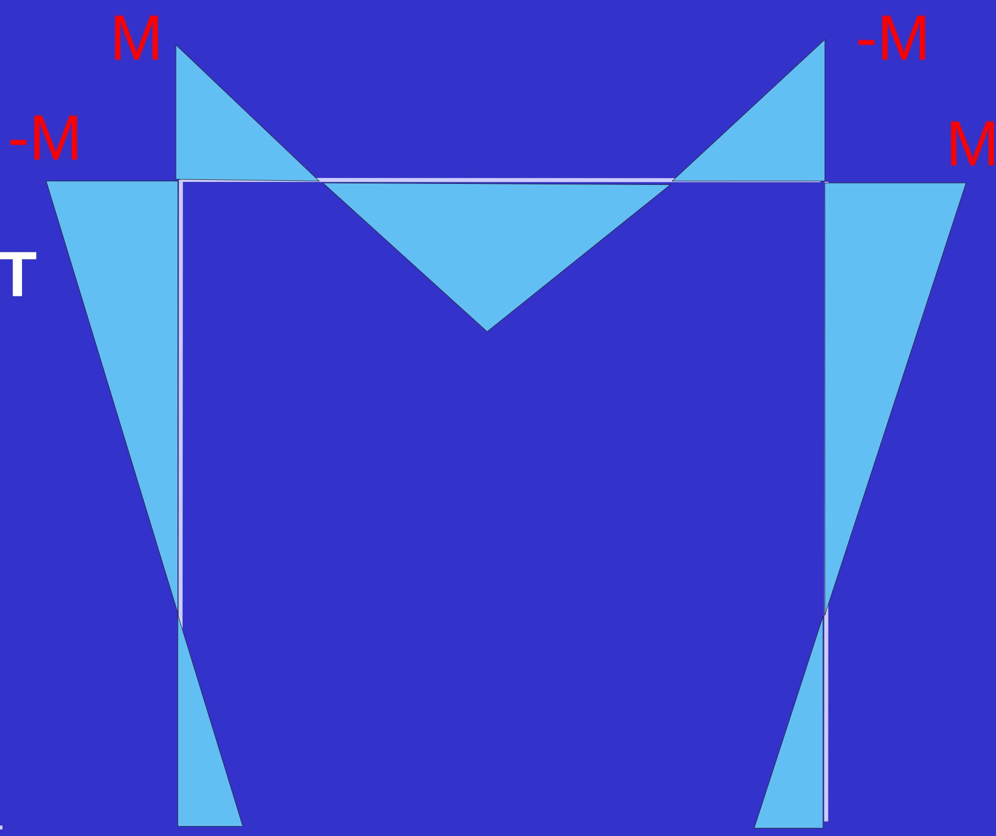




Bending moment diagram

Some examples: 4





To analyse statically indeterminate plane-frame structures

Write down expressions for the moments at each end of each member in a frame

Each expression will contain a fixed-end moment and a slope-deflection moment.

Impose equilibrium conditions:
i.e. at any joint $\Sigma M = 0$. At any pinned end, $M = 0$.

Impose joint rotation compatibility, i.e. a rigid joint which rotates must result in the same rotations for all member ends at that point

This gives a series of simultaneous equations for the rotations and deflections of the ends…
which can be solved.

Remember what we have already learnt

The total moments can be obtained by adding the fixed-end moments and the displacement moments.

$$M_{AB} = M_{FAB} + M_{DAB}$$

$$M_{BA} = M_{FBA} + M_{DBA}$$

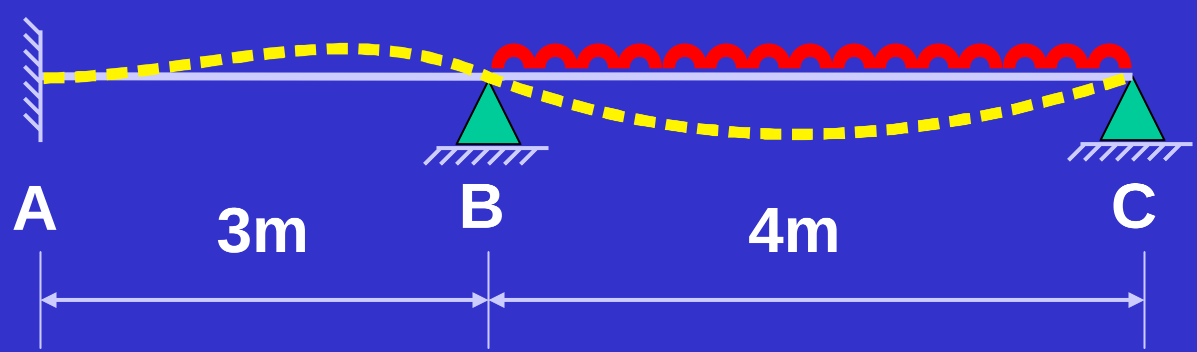
The **fixed end moments** can be found from equilibrium and compatibility (or by looking up the solution to simple cases)

The **displacement moments** are function of the end rotations and relative displacements of the ends of a beam

$$M_{DAB} = \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\delta}{L}\right)$$

$$M_{DBA} = \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\delta}{L}\right)$$

Example problem 1:



$EI = 420 \times 10^3$ kN/m^2 for AB & BC

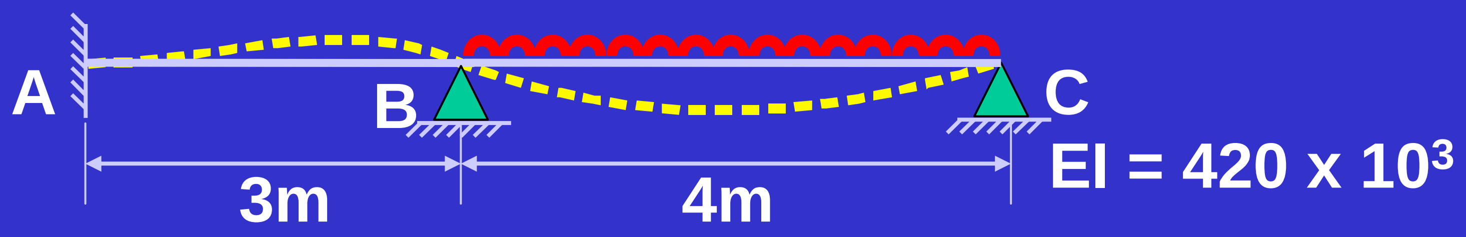
Unknown rotations: θ_B & θ_c

Write down fixed-end moments:

$M_{FAB} = M_{FBA} = 0$

$M_{FBC} = 20 \times 4^2/12 \quad = 26.7$ kNm

$M_{FCB} = -26.7$ kNm



Write down slope-deflection equations:

$\theta_A = 0$

For AB: $\delta = 0$ For BC: $\delta = 0$

$$M_{DAB} = 280\times10^{3}\left(\theta_B\right)$$

$$M_{DBA} = 560\times10^{3}\left(\theta_B\right)$$

$$M_{DBC} = 210\times10^{3}\left(2\theta_B + \theta_C\right)$$

$$M_{DCB} = 210\times10^{3}\left(2\theta_C + \theta_B\right)$$

Example problem 1:

Now recall that: $M_{BA} = M_{FBA} + M_{DBA}$

and $M_{BC} = M_{FBC} + M_{DBC}$

For equilibrium at joint B: $M_{BA} + M_{BC} = 0$

Hence: $0 = 560 \times 10^3\, \theta_B + 210 \times 10^3 (2\theta_B + \theta_C) + 26.7$

$980\, \theta_B + 210\, \theta_C = -0.0267$

For equilibrium at joint C: $M_{CB} = 0$

Hence: $0 = 210 \times 10^3 (2\theta_C + \theta_B) - 26.7$

$210\, \theta_B + 420\, \theta_C = 0.0267$

Example problem 1:

$$980\,\theta_B + 210\,\theta_C = -0.0267$$

$$210\,\theta_B + 420\,\theta_C = 0.0267$$

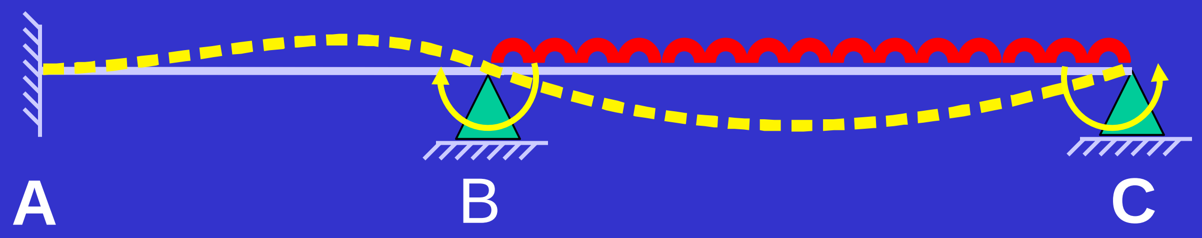
Solving the simultaneous equations gives

$$\theta_B = -45.7 \times 10^{-6} \text{ radians}$$
$$\theta_C = 86.4 \times 10^{-6} \text{ radians}$$

Note: θ_B **is -ve hence**

θ_C is +ve **hence**

This agrees with expected rotations



Example problem 1:

$$M_{AB} = -12.8kNm$$

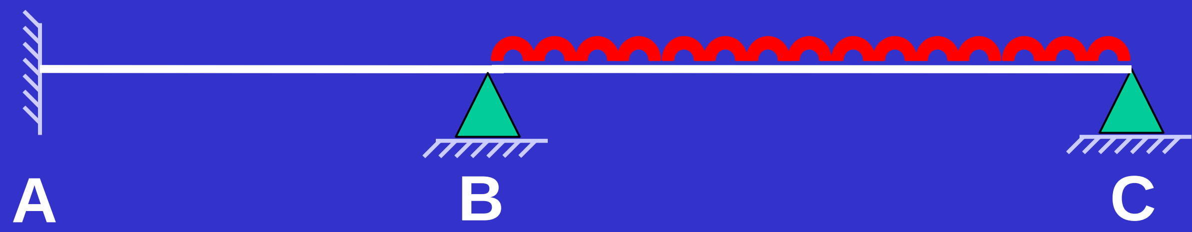
$$M_{BA} = -25.6kNm$$

$$M_{BC} = 25.6kNm$$

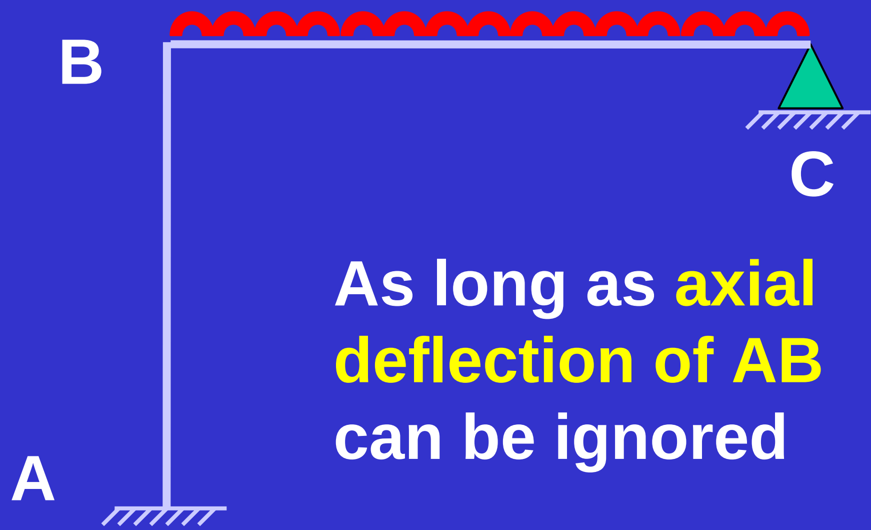
$$M_{CB} = 0$$

Example problem 1:

Note that the problem:

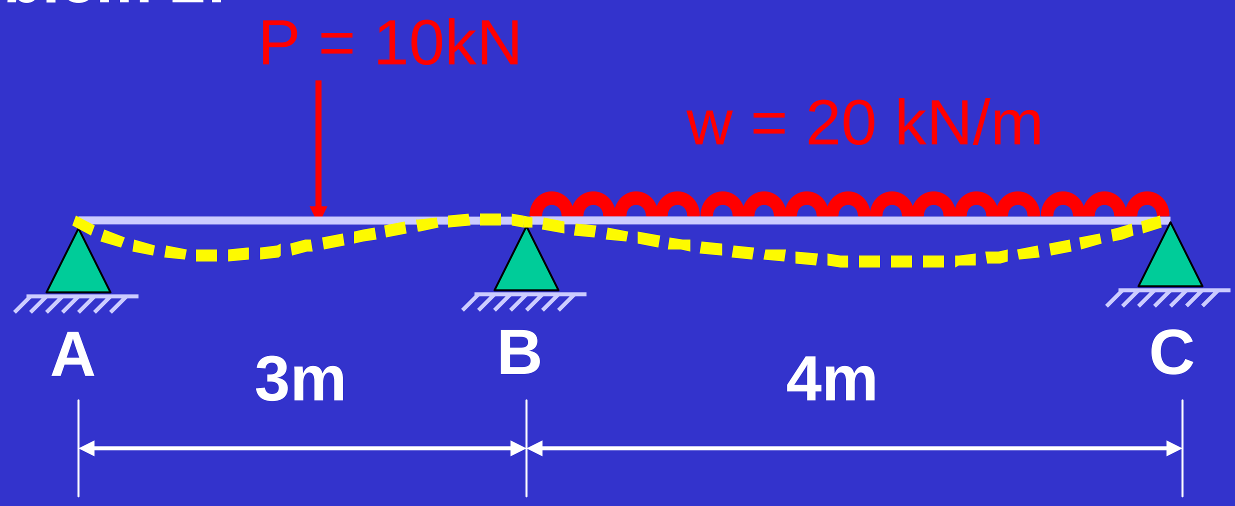


Is the same problem as:



As long as axial deflection of AB can be ignored

Example problem 2:



$EI = 500 \times 10^3\ kN/m^2$ for AB & BC

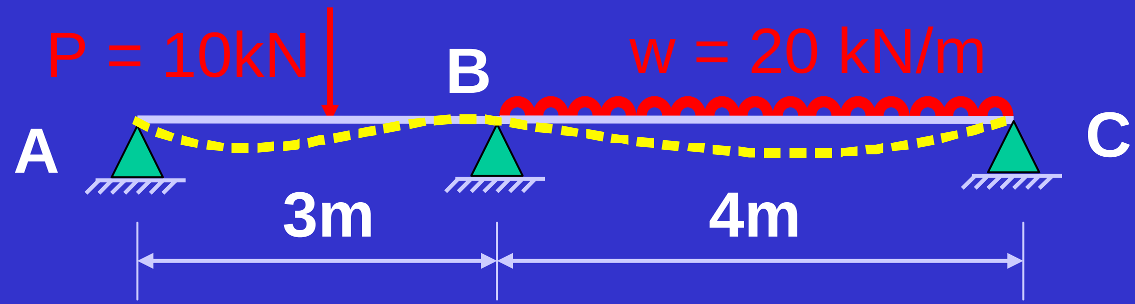
Unknown rotations: θ_A, θ_B & θ_c Note: θ_A θ_C

θ_B ???

Write down fixed-end moments:

$M_{FAB} = 10 \times 3/8 = 3.75\ kNm = -M_{FBA}$

$M_{FBC} = 26.7\ kNm = -M_{FCB}$



EI = 500 x 10[3]

Write down slope-deflection equations:

For AB: δ = 0   For BC: δ = 0

$$M_{DAB} = 333 \times 10^3 (2\theta_A + \theta_B)$$

$$M_{DBA} = 333 \times 10^3 (2\theta_B + \theta_A)$$

$$M_{DBC} = 250 \times 10^3 (2\theta_B + \theta_C)$$

$$M_{DCB} = 250 \times 10^3 (2\theta_C + \theta_B)$$

Now recall that: $M_{AB} = M_{FAB} + M_{DAB}$

and $M_{BA} = M_{FBA} + M_{DBA}$

$M_{BC} = M_{FBC} + M_{DBC}$ **…and so on**

For equilibrium at joint A: $M_{AB} = 0$

Hence: $0 = 3.75 + 333 \times 10^3 \quad (2\theta_A + \theta_B)$

$$666\,\theta_A + 333\,\theta_B = -0.00375$$

For equilibrium at joint B: $M_{BA} + M_{BC} = 0$

Hence: $0 = -3.75 + 333\text{x}10^3(2\theta_B + \theta_A) + 26.7 + 250 \times 10^3(2\theta_B + \theta_C)$

$$333\theta_A + 1166\,\theta_B + 250\,\theta_C = 0.02295$$

For equilibrium at joint C: $M_{CB} = 0$

Hence: $0 = -\,26.7 + 250 \times 10^{3}(2\theta_C + \theta_B)$

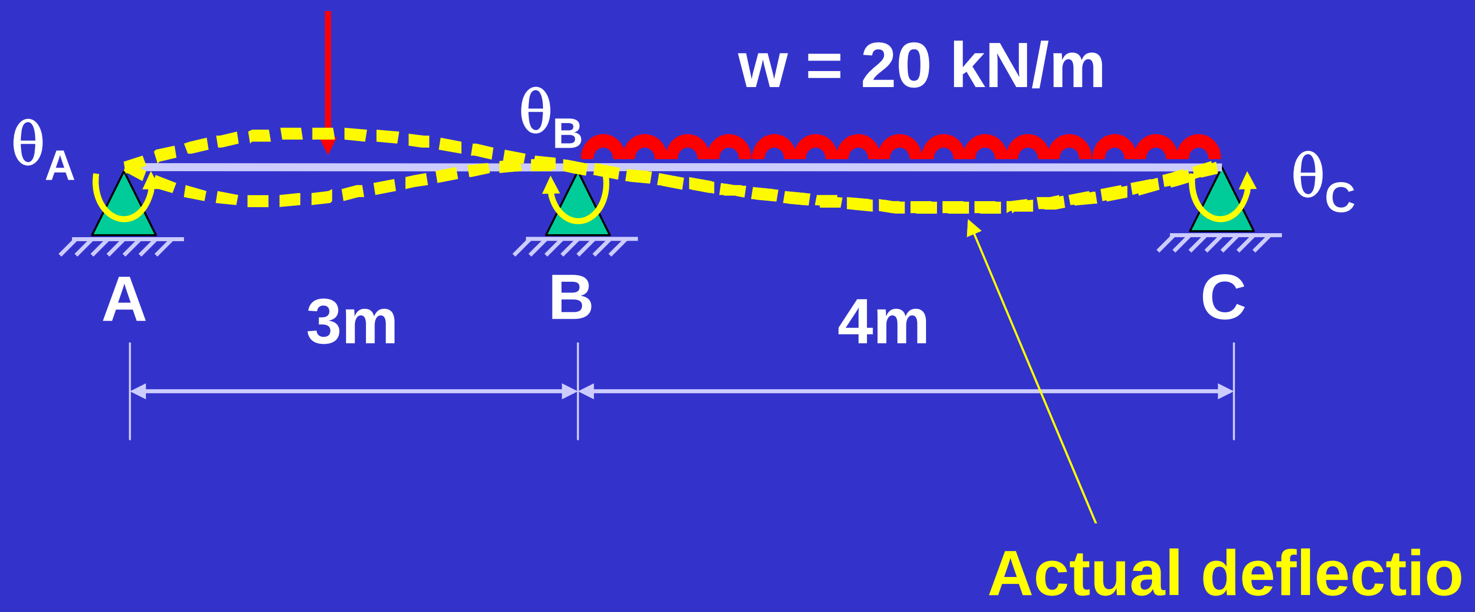
$$250\,\theta_B + 500\,\theta_C = 0.0267$$

Solving the simultaneous equations gives

$\theta_A = 13.9 \times 10^{-6}$ radians θ_A

$\theta_B = -39 \times 10^{-6}$ radians θ_B

$\theta_C = 73 \times 10^{-6}$ radians θ_C



w = 20 kN/m
θA
θB
θC
A
B
C
3m
4m
Actual deflectio

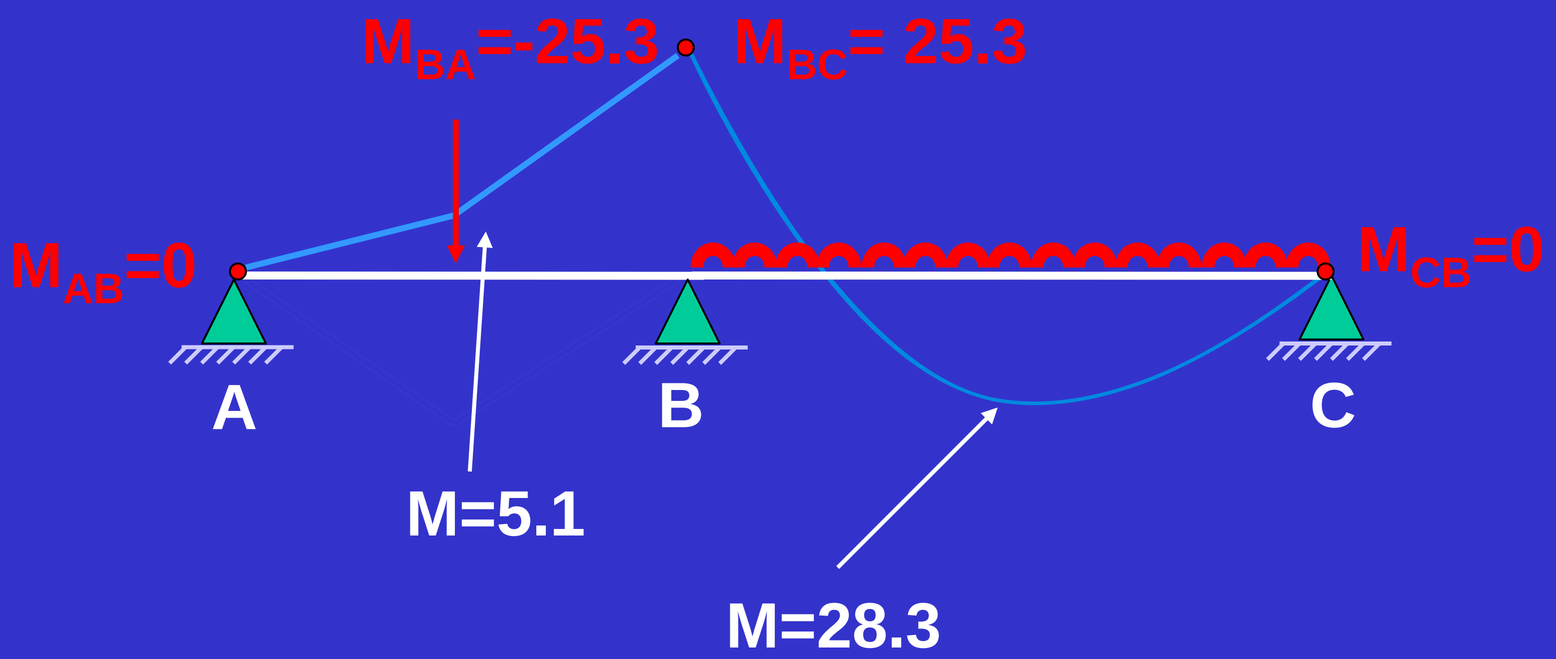
$$M_{AB} = 0\,kNm$$

$$M_{BA} = -25.3\,kNm$$

$$M_{BC} = 25.3\,kNm$$

$$M_{CB} = 0\,kNm$$

Example problem 2:



Tutorial 2: Slope-deflection

Now it's your turn…! Tutorial 2